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# SOME $I$-CONVERGENCE OF DIFFERENCE SEQUENCES OF FUZZY REAL NUMBERS

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**Abstract.** In this article we introduce some $I$-convergent difference sequence spaces of fuzzy real numbers defined by modulus function and study their different properties like completeness, solidity, symmetricity etc.

**1. Introduction.**The notion of $I$-convergence of real valued sequence was studied at the initial stage by Kostyrko, Šalát and Wilczyński (2000–2001) which generalizes and unifies different notions of convergence of sequences. The notion was further studied by Šalát, Tripathy and Ziman (2004).

The notion of fuzzy sets was introduced by Zadeh (1965). After that many authors have studied and generalized this notion in many ways, due to the potential of the introduced notion. Also it has wide range of applications in almost all the branches of studied in particular science, where mathematics is used. It attracted many workers to introduce different types of fuzzy sequence spaces.

Bounded and convergent sequences of fuzzy numbers were studied by Matloka (1986). Later on sequences of fuzzy numbers have been studied by Kaleva and Seikkala (1984), Tripathy and Sarma (Tripathy and Das, 2008 and Zadeh, 1965) and many others.

A function $f : [0, \infty) \to [0, \infty)$ is called a modulus function if

(a) $f(x) = 0$ if and only if $x = 0$

(b) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$

(c) $f$ is increasing

(d) $f$ is continuous from the right at 0.

Kizmaz (1981) defined the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$,$c_0(\Delta)$ for crisp sets as follows:

$$Z\,(\Delta) = \{X = (X_k) : \Delta X_k \in Z\}$$

where $Z = \ell_\infty, c$ and $c_0$.

**2. Definitions and Background.** Let $X$ be a non-empty set, then a non-void class $I \subseteq 2^X$ (power set of $X$) is called an ideal if $I$ is additive (i.e. $A, B \in I \Rightarrow A \cup B \in I$) and hereditary (i.e. $A \in I$ and $B \subseteq A \Rightarrow B \in I$). An ideal $I \subseteq 2^X$ said to be non-trivial if $I \neq 2^X$. A non-trivial ideal $I$ is said to be admissible if $I$ contains every finite subset of $N$. A non-trivial ideal $I$ is said to be maximal if there does not exist any non-trivial ideal $J \neq I$ containing $I$ as a subset.

Let $X$ be a non-empty set, then a non-void class $F \subseteq 2^X$ is said to be a filter in $X$ if $\phi \notin F$; $A, B \in F \Rightarrow A \cap B \in F$ and $A \in F$, $A \subseteq B \Rightarrow B \in F$. For any ideal $I$ there is a filter $\Psi(I)$ corresponding to $I$, given by

$\Psi(I) = \{K \subseteq N : N \backslash K \in I\}$.

Let $D$ denote the set of all closed and bounded intervals $X = [a_1, b_1]$ on the real line $R$. For $X = [a_1, b_1] \in D$ and $Y = [a_2, b_2] \in D$, define

$$d(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|)$$

It is known that $(D, d)$ is a complete metric space.

A fuzzy real number $X$ is a fuzzy set on $R$ i.e. a mapping $X : R \to L(= [0, 1])$ associating each real number $t$ with its grade of membership $X(t)$.

The $\alpha$-level set $X^\alpha$ of a fuzzy real number $X$ for $0 < \alpha \leq 1$, defined as

$$X^\alpha = \{t \in R : X(t) \geq \alpha\}$$

A fuzzy real number $X$ is called convex, if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number $X$ is called normal.

A fuzzy real number $X$ is said to be upper semi-continuous if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, is open for all $a \in L$ in the usual topology of $R$.

The set of all upper semi-continuous, normal, convex fuzzy number is denoted by $L(R)$.

The absolute value $|X|$ of $X \in L(R)$ is defined as

$$|X|(t) = \left\{ \begin{array}{cc} \max\{X(t), X(-t)\}, & \text{if } t > 0 \\ 0, & \text{if } t < 0 \end{array} \right\}$$

Let $\bar{d} : L(R) \times L(R) \to R$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha)$$

Then $\bar{d}$ defines a metric on $L(R)$.

A sequence $(X_k)$ of fuzzy real number is said to be convergent to the fuzzy real number $X_0$, if for every $\varepsilon > 0$, there exists $n_0 \in N$ such that $\bar{d}(X_k, X_0) < \varepsilon$ for all $k \geq n_0$.

A fuzzy real valued sequence space $E^F$ is said to be solid if $(Y_k) \in E^F$ whenever $(X_k) \in E^F$ and $|Y_k| \le |X_k|$, for all $k \in N$.

A sequence $(X_k)$ of fuzzy numbers is said to be $I$-convergent if there exists a fuzzy number $X_0$ such that for all $\varepsilon > 0$, the set $\{n \in N : \bar{d}(X_k, X_0) \ge \varepsilon\} \in I$.

We write $I - \lim X_k = X_0$).

A sequence $(X_k)$ of fuzzy numbers is said to be $I$-bounded if there exists a real number $\mu$ such that the set $\{k \in N : \bar{d}(X_k, \bar{0}) > \mu\} \in I$.

Throughout $c^{I(F)}$, $c_0^{I(F)}$ and $\ell_\infty^{I(F)}$ denote the spaces of fuzzy real-valued $I$-convergent, $I$-null and a $I$-bounded sequences respectively.

We define the following classes of sequences:

$$(c^I)^F(f,\Delta) = \Big\{(X_k) : \Big\{k : f\Big(\frac{\bar{d}(\Delta X_k, L)}{r}\Big) \ge \varepsilon, \text{ for some } r > 0 \text{ and } L \in R(I)\Big\} \in I\Big\}$$

$$(c_0^I)^F(f,\Delta) = \Big\{(X_k) : \Big\{k : f\Big(\frac{\bar{d}(\Delta X_k, \bar{0})}{r}\Big) \ge \varepsilon, \quad \text{for some } r > 0\Big\} \in I\Big\}$$

Also we define

$$(m^I)^F(f,\Delta) = (c^I)^F(f,\Delta) \cap \ell_\infty^F(f,\Delta),$$
$$(m_0^I)^F(f,\Delta) = (c_0^I)^F(f,\Delta) \cap \ell_\infty^F(f,\Delta)$$

## 3. Main Results

THEOREM 3.1 *The spaces* $(m^I)^F(f,\Delta)$ *and* $(m_0^I)^F(f,\Delta)$ *are complete metric spaces with respect to the metric given by*

$$g(X,Y) = \bar{d}(X_1, Y_1) + \inf\Big\{r > 0 : \sup_k f\Big(\frac{\bar{d}(\Delta X_k, \Delta Y_k)}{r}\Big) \le 1\Big\}$$

*Proof:* Let $(X^n)$ be a Cauchy sequence in $(m^I)^F(f,\Delta)$, where $X^n = (X_k^n)$

Let $\varepsilon > 0$ be given. For a fixed $x_0 > 0$, choose $r > 0$ such that $f\big(\frac{rx_0}{3}\big) \ge 1$ and $m_0 \in N$ such that $g(X^n, X^m) < \frac{\varepsilon}{rx_0}$ for all $n, m \ge m_0$

By definition of $g$ we have, $\bar{d}\big(X_1^m, X_1^n\big) < \varepsilon$

Thus $(X_1^m)$ is a Cauchy sequence of fuzzy real numbers and so $\lim_m X_1^m$ exist.

Again

$$f\left(\frac{\bar{d}(\Delta X_k^m, \Delta X_k^n)}{g(X^m, X^n)}\right) \le 1 \le f\left(\frac{rx_0}{3}\right)$$
$$\Rightarrow \quad \bar{d}(\Delta X_k^m, \Delta X_k^n) < \frac{\varepsilon}{3}$$

Thus $(\Delta X_k^m)$ is a Cauchy sequence of fuzzy real numbers and so $\lim_m \Delta X_k^m = \Delta X_k$ exist.

Moreover using the existence of $\lim_m X_1^m$ we can conclude that $\lim_m X_k^m$ exist.

Using continuity of $f$, $f\left(\dfrac{\bar{d}(\Delta X_k^m, \Delta X_k)}{r}\right) \le 1$

Taking infimum of such $r$'s we get $g(X^n, X) < \frac{\varepsilon}{rx_0} < \varepsilon$ for all $n \ge m_0$.

Thus $(X^n)$ converges to $X$.

Since $X^m, X^n \in (m^I)^F(f, \Delta)$ so there exist fuzzy numbers $Y_m$, $Y_k$ such that

$$\begin{aligned} A &= \Big\{k \in N : f\Big(\frac{\bar{d}(\Delta X_k^m, Y_k)}{r}\Big) < f\Big(\frac{\varepsilon}{3r}\Big)\Big\} \in \Psi \\ &= \Big\{k \in N : \bar{d}(\Delta X_k^m, Y_k) < \Big(\frac{\varepsilon}{3}\Big)\Big\} \in \Psi \\ B &= \Big\{k \in N : \bar{d}(\Delta X_k^m, Y_m) < \Big(\frac{\varepsilon}{3}\Big)\Big\} \in \Psi \end{aligned}$$

Now $A \cap B \in$ and let $k \in A \cap B$.

Then

$$\begin{aligned} \bar{d}(Y_k, Y_m) &\le \bar{d}(Y_k, \Delta X_k^n) + \bar{d}(\Delta X_k^n, \Delta X_k^m) + \bar{d}(\Delta X_k^m, Y_m) \\ &< \epsilon \text{ for all } n, m \ge m_0. \end{aligned}$$

Thus $(Y_k)$ is a Cauchy sequence of fuzzy real numbers. So there exists a fuzzy real number $Y$ such that $\lim Y_k = Y$. To show that $I$-$\lim \Delta X_k = Y$.

This follows from above inequalities as

$$\begin{aligned} \bar{d}(\Delta X_k, Y) &\le \bar{d}(\Delta X_k, \Delta X_k^m) + \bar{d}(\Delta X_k^m, Y_k) + \bar{d}(\Delta Y_k, Y) \\ &< \eta. \end{aligned}$$

Thus $I$-$\lim X_k = Y$. Thus $(X_k) \in (m^I)^F(f, \Delta)$.

THEOREM 3.2 *The sequence spaces* $(c_0^I)^F(f, \Delta)$, $(m^I)^F(f, \Delta)$ *and* $(m_0^I)^F(f, \Delta)$ *are solid.*

*Proof:* We prove the result for $(c_0^I)^F(f, \Delta)$. For the other spaces the result can be proved similarly.

Let $(X_k) \in (c_0^I)^F(f, \Delta)$ and $(Y_k)$ be such that $|Y_k| \le |X_k|$, for all $k \in N$. Then for given $\varepsilon > 0$, $A = \Big\{k \in N : f\Big(\frac{\bar{d}(\Delta X_k, \bar{0})}{r}\Big) \ge \varepsilon$, for some $r > 0\Big\} \in I$

Since $f$ is increasing, $B = \Big\{k \in N : f\Big(\frac{\bar{d}(\Delta Y_k, \bar{0})}{r}\Big) \ge \varepsilon$, for some $r > 0\Big\} \subset A$

Thus $B \in I$ and so $(Y_k) \in (c_0^I)^F(f, \Delta)$. Hence $(c_0^I)^F(f, \Delta)$ is solid.

PROPERTY 3.3 *The sequence spaces* $(c_0^I)^F(f,\Delta)$, $(c^I)^F(f,\Delta)$, $(m^I)^F(f,\Delta)$ *and* $(m_0^I)^F(f,\Delta)$ *are not convergence free.*

For this result consider the following example.

EXAMPLE 3.2 *Let* $I = I_\delta$ *and* $f(x) = x$. *Consider the sequence* $(X_k)$ *defined as follows*:

For $k \neq i^2,\ i \in N$

$$X_k(t) = \begin{cases} 1, & \text{for } 0 \le t \le k^{-1} \\ 0, & \text{otherwise} \end{cases}$$

And for $k = i^2,\ i \in N,\quad X_k(t) = \bar{0}$.

Then for $\alpha \in (0,1]$ we have,

$$[X_k]^\alpha = \begin{cases} [0,0], & \text{if } k = i^2 \\ [0,k^{-1}] & \text{if } k \neq i^2 \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} [-(k+1)^{-1},0], & \text{for } k = i^2 \\ \left[0,k^{-1}\right], & \text{for } k = i^2 - 1\ (i \neq 1) \\ \left[-(k+1)^{-1},k^{-1}\right], & \text{otherwise} \end{cases}$$

Hence $\Delta X_k \to \bar{0}$ as $k \to \infty$. Thus $(X_k) \in (c_0^I)^F(f,\Delta) \subset (c^I)^F(f,\Delta)$

Let $(Y_k)$ be another sequence such that:

$$Y_k(t) = \begin{cases} 1, & \text{if } 0 \le t \le k \\ 0, & \text{otherwise} \end{cases}$$

And for $k = i^2,\ i \in N,\quad Y_k(t) = \bar{0}$.

Now for all $\alpha \in (0,1]$ we have,

$$[Y_k]^\alpha = \begin{cases} [0,0], & \text{if } k = i^2 \\ [0,k] & \text{if } k \neq i^2 \end{cases}$$

and

$$[\Delta Y_k]^\alpha = \begin{cases} [-(k+1),0], & \text{for } k = i^2 \\ [0,k], & \text{for } k = i^2 - 1\ (i \neq 1) \\ [-(k+1),k], & \text{otherwise} \end{cases}$$

This implies $(Y_k) \notin (c_0^I)^F(f,\Delta) \subset (c^I)^F(f,\Delta)$

Hence $(c_0^I)^F(f,\Delta)$, $(c^I)^F(f,\Delta)$ are not convergence free. Similarly the other spaces are also not convergence free.

THEOREM 3.4 $Z(f_1, \Delta) \subseteq Z(f_2 o f_1, \Delta)$ *for* $Z = (c^I)^F,\ (c_0^I)^F$ *and* $(\ell_\infty^I)^F$.

*Proof:* Let $Z = (c^I)^F$ and $(X_k) \in (c^I)^F(f_1, \Delta)$

Then
$$\left\{k : f_1\left(\frac{\bar{d}(\Delta X_k, L)}{r}\right) \geq \varepsilon, \quad \text{for some} \quad r > 0\right\} \in I$$

Since $f_2$ is continuous, so for $\varepsilon > 0$ there exist $\eta > 0$ such that $f_2(\varepsilon) = \eta$.

The result follows from
$$f_2\left(f_1\left(\frac{\bar{d}(\Delta X_k, L)}{r}\right)\right) \geq f_2(\varepsilon) = \eta$$

THEOREM 3.5 $Z(f, \Delta) \subseteq (\ell_\infty^I)^F(f, \Delta)$ *for* $Z = (c^I)^F,\ (c_0^I)^F$

*Proof:* The proofs are obvious.

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