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Some generalized sequence spaces operated by a modulus function

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Abstract

In this article we study different properties of the sequence space $m_F(f, \phi, p)$, $0 < p < 1$. Some inclusion results will be established among the introduced sequence spaces and existing sequence spaces. We study the property of solidity.

Key words : Modulus function, Monotone space, fuzzy sequence, completeness, solidity
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1. Introduction

The sequence space $m(\phi)$ was introduced by Sargent[15], who studied its different properties and obtained its relations with the spaces ℓ^p and ℓ^∞. Later on the notion was further investigated and linked with summability theory by Sarma[16], Tripathy and Sen[18] and many others.

Spaces of sequences of fuzzy numbers were studied by Matloka[9], Nuray and Savas[13] and many others.

The notion of modulus function was introduced by Nakano[11]. Later on different sequence spaces were defined by using modulus function and their different properties were investigated by Ruckle[14], Maddox[8], Bilgin[4] and many others.

Throughout the article w^F and $(\ell_\infty)_F$ denote the spaces of *all* and *bounded* sequences of fuzzy numbers, respectively.

Let P_s denote the class of all subsets of N, the set of natural numbers, those do not contain more than s elements. Throughout $\{\phi_n\}$ represents a non-decreasing sequence of real numbers such that $n\phi_{n+1} \leq (n+1)\,\phi_n$, for all $n \in N$.

The class of these sequences $\{\phi_n\}$ is denoted by Φ.

The sequence space $m(\phi)$ introduced by Sargent[15] is defined as :

$$m(\phi) = \{(x_k) \in w : \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s}\sum_{k\in\sigma}|x_k| < \infty\},$$

which becomes a Banach space, normed by

$$|x|_{m(\phi)} = \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\varphi_s}\sum_{k\in\sigma}|x_k|.$$

Let D denote the set of all closed and bounded intervals $X = [a_1, a_2]$ on R, the real line. For X, Y $\in$ D we define

$$d(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|),$$

where $X = [a_1, a_2]$ and $Y = [b_1, b_2]$. It is known that (D, d) is a complete metric space.

A fuzzy real number X is a fuzzy set on R, *i.e.* a mapping $X:R\rightarrow I$ $(=[0, 1])$ associating each real number t with its grade of membership X(t).

A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) = \min\{X(s), X(t)\}$, where $s<t<r$.

If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

A fuzzy real number X is said to be *upper-semi continuous* if, for each $\varepsilon > 0$, $X^{-1}([0, a+\varepsilon))$, for all $a \in I$ is open in the usual topology of R.

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by R(I) and throughout the article, by a fuzzy real number we mean that the number belongs to R(I).

The α - level set $[X]^\alpha$ of the fuzzy real number X, for $0 < \alpha \leq 1$, defined as $[X]^\alpha = \{t \in R : X(t) \geq \alpha\}$. If $\alpha = 0$, then it is the closure of the strong 0-cut.

The set R of all real numbers can be embedded in R(I). For $r \in R$, $\bar{r} \in R(I)$ is defined by

$$\bar{r}(t)\begin{cases} 1, & \text{for } t = r, \\ 0, & \text{for } t \neq r \end{cases}$$

The *absolute* value, $|X|$ of $X \in R(I)$ is defined by (see for instance Kaleva and Seikkala[6])

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

A fuzzy real number X is called non-negative if $X(t) = 0$, for all $t<0$. The set of all non-negative fuzzy real numbers is denoted by $R^*(I)$.

Let $\bar{d} : R(I) \times R(I) \to R$ be defined by

$$\bar{d}(X, Y) = \sup_{0\leq\alpha\leq 1} d\left([X]^\alpha, [Y]^\alpha\right)$$

Then $\bar{d}$ defines a metric on R(I).

The additive identity and multiplicative identity in R(I) are denoted by $\bar{0}$ and $\bar{1}$ respectively.

2. Definition and Preliminaries

Definition. A function $f : [0, \infty) \to [0, \infty)$ is called a *modulus* if

(a) $f(x) = 0$ if and only if $x = 0$
(b) $f(x + y) \leq f(x) + f(y)$, for $x \geq 0$, $y \geq 0$.
(c) f is increasing.
(d) f is continuous from the right at 0.

Hence f is continuous everywhere in $[0, \infty)$.

We define the following sequence space

$$m_F(f, \phi, p) = \left\{(X_k) \in w^F : \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} [f(\bar{d}(X_k, \bar{0}))]^p < \infty\right\}$$

Definition. A sequence space E is said to be *solid* (or *normal*) if $(Y_k) \in E$, whenever $|Y_k| \leq |X_k|$, for all $k \in N$, for some $(X_k) \in E$.

Definition. Let $K = \{k_1 < k_2 < k_3 \ldots\} \subseteq N$ and E be a sequence space. A *K-step space* of E is a sequence space $\lambda_K^{E^F} = \{(X_{k_n}) \in w^F : (X_n) \in E\}$.

Definition. A *canonical pre-image of a sequence* $(X_{k_n}) \in \lambda_K^{E^F}$ is a sequence $(Y_n) \in w^F$ defined as follows:

$$Y_n = \begin{cases} X_n, & \text{if } n \in K, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

A *canonical pre-image of a step space* λ_K^E is a set of canonical pre-images of all elements in λ_K^E, *i.e.*, Y is in canonical pre-image λ_K^E if and only if Y is canonical pre-image of some $X \in \lambda_K^E$.

Remark 1. If a sequence space E is solid then E is monotone.

Main Results

Theorem 3.1. The set $m_F(f, \phi, p)$ is a complete linear metric space, with respect to the metric g defined by

$$g(X, Y) = \sup_{s\geq 1,\ \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} [f(\bar{d}(X_k, Y_k))]^p$$

Proof. Since the linearity of $m_F(f, \phi, p)$ with respect to the co-ordinate wise addition and scalar multiplication is trivial, we omit the details.

Theorem 3.2. Let $0<p<1$, then $\ell_F^p(f) \subseteq m_F(f, \phi, p)$ for all sequences (ϕ_s) in Φ. Further $\ell_F^p(f) = m_F(f, \phi, p)$ if and only if $(\phi_n) \in c$.

Proof. We have $\ell_F^p(f) = m_F(f, \phi, p)$ when $\phi_n = 1$ for all $n \in N$. Since (ϕ_n) is increasing, $\sup_{s\geq 1} \frac{1}{\varphi_s} = \frac{1}{\varphi_1} < \infty$.

Hence $\ell_F^p(f) \subseteq m_F(f, \phi, p)$ for all sequences (ϕ_s) in Φ. Converse part follows from the Theorem 3.2 taking $\psi_n = 1$ for all $n \in N$.

Theorem 3.3. Let $0 < t < p < 1$. Then

(a) $m_F(f, \phi, p) \subseteq m_F(f, \phi, t)$.

(b) $m_F(f, \phi, p) \subseteq m_F(f, \psi, t)$, if and only if $\sup_{s \geq 1} \frac{\varphi_s}{\psi_s} < \infty$.

Proof. Let $(X_k) \in m_F(f, \phi, p)$.

Then $\sup_{s\geq 1,\ \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} [f(\bar{d}(X_k, \bar{0})]^p < \infty$

Now $0 < t < p < 1$.

Thus,

$$\sup_{s\geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} [f(\bar{d}(X_k, \bar{0})]^t < \sup_{s\geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} [f(\bar{d}(X_k, \bar{0})]^p < \infty$$

Hence

$(X_k) \in m_F(f, \phi, t)$.

Part (b) follows from the Theorem 3.3 and part (a) of this result.

Proposition 3.4. $m_F(f, \phi, p) \subseteq (\ell_\infty)_F(f)$.

Theorem 3.5. The space $m_F(f, \phi, p)$ is solid and hence monotone.

Proof. Let $(X_k) \in m_F(f, \phi, p)$ and (Y_k) be a sequence such that

$|Y_k| \leq |X_k|$ for all $k \in N$. Then the result follows from the following inequality :

$(f(\bar{d}(Y_k, \bar{0}))p < (f(\bar{d}(X_k, \bar{0}))^p$ for all $k \in N$.

The rest follows from the Remark 1.

References

[1] Y. Altin, M. Et and R. Colak, Lacunary statistical and lacunary strongly convergence of generalized difference sequences of fuzzy numbers; *Computers and Math. Appl.*, **52**, (2006) 1011-1020.

[2] H. Altnok, Y. Altin and M. Et, Lacunary almost statistical convergence of fuzzy numbers; *Thai J. Math.* **2**(2), (2004) 265-274.

[3] M. Basarir and M. Mursaleen, Some difference sequence spaces of fuzzy numbers; *J. Fuzzy Math.* **11**(3), (2003) 1-7.

[4] T. Bilgin, The sequence space on seminormed spaces; *Bull. Cal. Math. Soc.* **86**, (1994) 295-304.

[5] M. Et, R. Colak and H. Altnok, On-statistical convergence of difference sequences of fuzzy numbers; *Infrom. Sci.* **176**(15), (2006) 2268-2278.

[6] O. Kaleva and S. Seikkala, On fuzzy metric spaces; *Fuzzy Sets and Systems;* **12**, (1984) 215-229.

[7] P.K. Kamthan and M. Gupta, Sequence spaces and Series; *Marcel Dekkar;* 1980.

[8] I.J. Maddox, Sequence spaces defined by a modulus; *Math. Proc. Camb. Phil. Soc.* 100, (1980) 161-166.

[9] M. Matloka, Sequences of fuzzy numbers; *BUSEFAL*, **28**, (1986) 28-37.

[10] M. Mursaleen and M. Basarir, On some new sequence spaces of fuzzy numbers; *Ind. J. Pure Appl. Math.* **34**(9), (2003) 1351-1357.

[11] H. Nakano, Concave modulars; *J. Math. Soc. Japan;* **5**, (1953), 29-49.

[12] S. Nanda, On sequences of fuzzy numbers, *Fuzzy sets and System,* **33**, (1989) 123-126.

[13] F. Nuray and E. Savas, Statistical convergence of fuzzy numbers; *Math. Slovaca*, 45(3), (1995) 269-273.

[14] W.H. Ruckle, FK spaces in which the sequence of coordinate vectors is bounded; *Canad. J. Math.;* **25**, (1973) 973-978.

[15] W.L.C. Sargent, Some sequence spaces related to the spaces; *Jour. London Math Soc.;* **35**, (1960) 161-171.

[16] B. Sarma, Some Sequence Spaces of Fuzzy Numbers Defined by Orlicz Function, *Acta Scientiarum Technology,* **37**(1), (2015) 85-87.

[17] B. Sarma, Some I-Convergence of Difference Sequences of Fuzzy Real Numbers; *Bull. Cal. Math. Soc.,* **109**(1), (2017) 69-74.

[18] B.C. Tripathy and M. Sen, On a new class of sequences related to the space; *Tamkang Jour. Math.*; **33**(2), (2002) 167-171.

[19] L.A. Zadeh, Fuzzy sets; *Inform and Control*, **8**, (1965) 338-353.